MATH 99R, SPRING 2019

WITH DANNY SHI

Contents

1. Lecture 1 – February 6, 2019

Homework is optional, and we'll be using the book *Category Theory in Context* by Emily Riehl. Emily was a grad student here a couple years ago and the book constitutes lectures from a course she taught about category theory.

Definition 1.1. A category $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}(\mathcal{C})$ and for each $X, Y \in \mathrm{Ob}(\mathcal{C})$ a collection of morphisms $\mathrm{Hom}(X, Y)$. If $f \in \mathrm{Hom}(X, Y)$, X is said to be its domain and Y its codomain. Each object $X \in \mathrm{Ob}(\mathcal{C})$ has a distinguished identity morphism $\mathbb{1}_X \in \mathrm{Hom}(X, X)$. Additionally, for each $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ there exists $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$. This data satisfies

- $\mathbb{1}_Y \circ f = f = f \circ \mathbb{1}_X$
- $(f \circ g) \circ h = f \circ (g \circ h)$

One observation is that each object is characterized by its identity morphism. What character theory says is that the morphisms are more important than the objects.

Example 1.2. (1) **Set** is the category whose objects are sets and morphisms are functions between sets. Implicitly, we also need to define $(f \circ g)x = f(g(x))$.
(2) **Top** is the category whose objects are topological spaces and morphisms are continuous maps. Composition is just composition of functions.
(3) **Grp** is the category whose objects are groups and whose morphisms are group homomorphisms. Historically, this example is the reason why the word 'morphism' appears in the definition of category.
(4) **Man** is the category whose objects are smooth manifolds and morphisms are smooth maps.
(5) $\mathbf{Mod}_R$ is the category of R-modules with morphisms module homomorphisms. When R is a field, this category is $\mathbf{Vect}_K$, and when R is $\mathbb{Z}$, it's the full subcategory **Ab** of **Grp**.

All of these categories are concrete because, informally speaking, their objects are sets with extra structure and their morphisms are just maps of sets which happen to satisfy some properties. They are contrasted with abstract categories, which do not consist of sets with extra structure.

Example 1.3. A group G can be witnessed as the one-object category BG whose morphisms are elements of G with composition rule simply the group operation.

The point of these examples is that you can make a category out of just about anything, and that morphisms need not be functions.

Example 1.4. A poset can be turned into a category whose objects are the elements of the set and where $\text{Hom}(X, Y) = *$ if $X < Y$ and it's empty otherwise. We take $* \circ * = *$, and it associates vacuously.

A set S can be turned into a category where the objects are elements of S and the only morphisms are the identity. There's no need to define composition.

"A poset is a thing where you got stuff." - Danny.

Definition 1.5. A category is small if its morphisms form a set.

Definition 1.6. A category is locally small if $\text{Hom}(X, Y)$ is a set $\forall X, Y \in \text{Ob}(\mathcal{C})$.

In everyday math, we say that two objects should be considered equivalent if they admit an isomorphism between each other. Category theory equips us with more general language to describe isomorphisms.

Definition 1.7. A morphism $f : X \to Y$ is an isomorphism if there exists a map $g : Y \to X$ such that $f \circ g = \mathbb{1}_Y$ and $g \circ f = \mathbb{1}_X$.

This is a beautiful definition. When there exists an isomorphism between X and Y, we write $X \cong Y$.

Some morphisms have special names: morphisms whose domain and codomain coincide are called endomorphisms, and endomorphisms which are isomorphisms are called automorphisms.

Example 1.8. Isomorphisms in **Top** are homeomorphisms. In **Man**, they're diffeomorphisms.

Definition 1.9. A groupoid is a category in which every morphism is an isomorphism.

Definition 1.10. A subcategory of $\mathcal{C}$ is a category whose objects are a subcollection of those of $\mathcal{C}$ and whose morphisms are a subcollection of the morphisms of $\mathcal{C}$.

Here's a fun fact: any category $\mathcal{C}$ contains a maximal groupoid whose objects are all the objects in $\mathcal{C}$ and whose morphisms are all the isomorphisms in $\mathcal{C}$.

Exercise 1.1 Show that a morphisms can have at most one inverse morphism.

Proof. Not surprisingly, this is the proof that inverses are unique in a monoid. After all, Hom sets are monoids.

$$\begin{aligned} f^{-1} &= f^{-1}\mathbb{1} \\ &= f^{-1}(ff_*^{-1}) \\ &= (f^{-1}f)f_*^{-1} \\ &= \mathbb{1}f_*^{-1} \\ &= f_*^{-1} \end{aligned}$$

□

Exercise 1.2 Say $f : X \to Y$ is a morphism and there exist a pair of morphisms $g, h : Y \to X$ such that $gf = \mathbb{1}_X$ and $fh = \mathbb{1}_Y$. Show that $g = h$ and f is an isomorphism.

Proof. It's the same trick as the previous exercise.

$$\begin{aligned} g &= g\mathbb{1} \\ &= g(fh) \\ &= (gf)h \\ &= \mathbb{1}h \\ &= h \end{aligned}$$

It follows immediately that f is an isomorphism. What we've proven is that when left- and right-inverses exist, they coincide and in fact form a two-side inverse. This exercise, instantiated in the category **Set**, appears frequently in introductory math classes. □

Tonight's homework is to read the exercises at the end of Chapter 1 in Emily's book. The goal is to learn about the Slice category.

2. Lecture 2 – February 8, 2019

Today we'll talk about the duality principle. Given some category, imagine an opposite category in which the objects are preserved by the morphisms' domains and codomains are reversed. Intuitively, dots stay the same and arrows are reversed.

Definition 2.1. For $\mathcal{C}$ a category, $\mathcal{C}^{op}$ is the opposite category of $\mathcal{C}$ where $\text{Ob}(\mathcal{C}^{op}) = \text{Ob}(\mathcal{C})$ and $f^{op} \in \text{Hom}_{\mathcal{C}^{op}}(X, Y) \iff f \in \text{Hom}_{\mathcal{C}}(X, Y)$. Identities get associated to identities, and $g^{op} \circ f^{op} := (f \circ g)^{op}$.

It turns out that $\mathcal{C}^{op}$ defines a category if and only if $\mathcal{C}$ is a category. Additionally, $\mathcal{C}$ and $\mathcal{C}^{op}$ contain the exact same information. In real life, it sometimes turns out that it's easier to prove a claim in $\mathcal{C}^{op}$ than in $\mathcal{C}$.

Example 2.2. Last time we learned how to construct categories from posets. The opposite of these categories are those in which arrows point from greater elements to lesser elements.

Example 2.3. Recall that BG is the one-object category created from the data of a group. In BG, composition corresponds to left multiplication, while in BG^{op} composition corresponds to left multiplication. In fact, for G^{op} the opposite group of G, $(BG)^{op}$ is $B(G^{op})$.

Notice that a theorem which shows that something holds for all categories $\mathcal{C}$ has shown that something holds for all $\mathcal{C}^{op}$ as well (since the $\mathcal{C}^{op}$ are just categories). Translating the statement from $\mathcal{C}^{op}$ to one in $\mathcal{C}$, you get something slightly different (for free!). In this way, every theorem has a dual theorem.

Proposition 2.4. *The following are equivalent:*

1. $f : X \to Y$ *is an isomorphism in* $\mathcal{C}$.
2. *For all objects* $c \in \mathcal{C}$, f *induces a bijection* $f_* : \text{Hom}(C, X) \to \text{Hom}(C, Y)$.
3. *For all objects* $c \in \mathcal{C}$, f *induces a bijection* $f^* : \text{Hom}(Y, C) \to \text{Hom}(X, C)$.

Proof. We'll show $(i) \iff (ii)$ and use duality to show $(i) \iff (iii)$. First suppose f is an isomorphism, with inverse $g : Y \to X$. Then there's another induced map $g_* : \text{Hom}(c, Y) \to \text{Hom}(c, X)$. $f_* \circ g_*$ and $g_* \circ f_*$ are the identity on $\text{Hom}(C, X)$ and $\text{Hom}(C, Y)$, so we have a bijection. So $(i) \implies (ii)$.

Now say f_* is a bijection $\text{Hom}(c, X) \to \text{Hom}(c, Y)$ for any $c \in \text{Ob}(\mathcal{C})$. Taking $c = Y$, the surjectivity of this map means that $\mathbb{1}_Y \in \text{Hom}(Y, Y)$ gets hit by some $g \in \text{Hom}(Y, X)$. This g satisfies $f \circ g = \mathbb{1}_Y$, so we have a right-inverse. Now take $c = X$. f_* sends $g \circ f$ to $g \circ f \circ g = f$. But it also sends $\mathbb{1}_X$ to f. Since it's an injection, $g \circ f = \mathbb{1}_X$. So g is a true inverse, and $(ii) \implies (i)$.

Now we use duality: the equivalence between (i) and (ii) holds for any categories. In particular, it holds for $\mathcal{C}^{op}$. $f^{op} : y \to x$ is an isomorphism in $\mathcal{C}^{op}$, so $f_*^{op} : \text{Hom}_{\mathcal{C}^{op}}(c, Y) \to \text{Hom}_{\mathcal{C}^{op}}(c, X)$ is a bijection. This gives us what we want.

□

So all isomorphisms can be detected at the level of bijections between Hom sets. Technically we assumed that our categories were locally small, but an analogous statements occurs for general categories, after some fussing around with classes.

Definition 2.5. Let $f : X \to Y$ be a morphism.

(1) f is a monomorphism if $fh = fk \implies h = k$ for any $h, k \in \mathrm{Hom}(-, X)$.
(2) f is an epimorphism if $hf = kh \implies h = k$ for any $h, k \in \mathrm{Hom}(X, -)$.

These definitions are dual in the sense that monomorphisms of $\mathcal{C}$ are epimorphisms of $\mathcal{C}^{op}$ and vice versa.

Example 2.6. In **Set**, monomorphisms are surjections and epimorphisms are injections.

Equivalently, monomorphisms induce injections on Hom sets under post-composition and epimorphisms under pre-composition.

Example 2.7. In the setup $X \xrightarrow{s} Y \xrightarrow{r} X$, s is often called a section and r a retraction. Sections are always monomorphisms and retractions are always epimorphisms. This is not too hard to see when considering the action on Hom sets. These kinds of mono/epimorphisms are sometimes called split mono/epimorphisms (i.e. when they have one-sided inverses).

It's pretty clear that isomorphisms are both monomorphisms and epimorphisms. Is the converse true? The answer is no. Consider the inclusion $\mathbb{Z} \to \mathbb{Q}$ in the category **Ring**. It's a monomorphism because it's an injection. It's an epimorphism because any map $\mathbb{Q} \to W$ is determined by where $\mathbb{Z} \subset \mathbb{Q}$ gets sent. But it's not an isomorphism because $\mathbb{Z}$ doesn't have enough inverses to receive an injection from $\mathbb{Q}$.

Proposition 2.8. *Monomorphisms compose. On the other hand, if gf is a monomorphism then f is a monomorphism (but we can't say anything about g).*

The dual statement to this is that epimorphisms compose and that if gf is an epimorphism, g is an epimorphism.

Exercises: In Emily's book, 1.2(ii), 1.2(iii), and 1.2(vi).

3. Lecture 3 – February 13, 2019

Here's a slogan for this class: "Any mathematical object should be considered with its structure-preserving morphisms." And one mathematical object we've encountered is that of the object - so what should morphisms between categories look like?

Definition 3.1. A functor between categories $F : \mathcal{C} \to \mathcal{D}$ consists of a map

- $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$
- $F : \mathrm{Hom}(c_1, c_2) \to \mathrm{Hom}(d_3), d_4)$

And these assignments satisfy the follow functoriality axioms:

- $F(\mathbb{1}_c) = \mathbb{1}_{F(c)}$
- $F(f \circ g) = F(f) \circ F(g)$

Example 3.2. The forgetful functor **Grp** $\to$ **Set** sends a group to its underlying set and group homomorphisms to the maps of sets which define them. Similarly, there's a forgetful functor **Top** $\to$ **Set**.

There can also be intermediate forgetful functors which forget some, but not all, of the structure in a category. For instance, there's a forgetful functor $\mathbf{Mod}_R \to$ **Ab** which forgets about scaling (the map $R \otimes M \to M$). There's also a forgetful functor from **Ring** $\to$ **Ab** which forgets about the operation of multiplication. It shouldn't be much of a surprise that none of these functors inject on objects.

Example 3.3. An extremely important functor in algebraic topology sends **Top** to **hTop**. It does nothing on objects and sends morphisms to their homotopy classes. Other functors in algebraic topology send $\mathbf{Top}_*$ to **Grp**, like π_i, homology, and cohomology. One of the first things you learn in algebraic topology is that these functors are homotopy invariant. This is witnessed in the fact that these maps factor through $\mathbf{hTop}_*$.

Example 3.4. A functor that is in some ways 'opposite' to the forgetful functor is the free functor, which sends **Set** to **Group**. A set gets sent to the group consisting of finite words with letters in that set. This quality of being opposite will be made more formal when we get to adjoints - the result is that the free and forgetful functors form an adjoint pair.

Theorem 3.5 (Brouwer's Fixed Point)**.** *Any continuous map $D^2 \to D^2$ has a fixed point.*

Proof. Suppose not. Then one can construct a continuous map $r : D^2 \to \partial D^2$ that sends x to to the point on the boundary which the ray with head $f(x)$ going through x hits. This map is continuous, and it acts as the identity on ∂D^2, since we're looking at a ray which hits x. So there are maps:

$$S^1 \xrightarrow{i} D^2 \xrightarrow{r} S^1$$

Note that $r \circ i = \mathbb{1}_{S^1}$. Moving to groups via π_1, and using the fact that pi_1 commutes with composition, there are maps

$$\pi_1(S^1) \to \pi_1(D^2) \to \pi_1(S^1)$$

whose composition is the identity. But D^2 has trivial fundamental group and S^1 doesn't, producing contradiction. $\square$

Definition 3.6. Covariant functors send $\mathrm{Hom}(c_1, c_2)$ to $\mathrm{Hom}(F(c_1), F(c_2))$. Contravariant functors send $\mathrm{Hom}(c_1, c_2)$ to $\mathrm{Hom}(F(c_2), F(c_1))$.

A contravariant functor is just a covariant functor out of $\mathcal{C}^{op}$.

Example 3.7. There's a dual functor in linear algebra $(-)^* : \text{Vect}_k^{op} \to \text{Vect}_k$, $V \mapsto V^* = \text{Hom}(V, K)$.

There's also a functor $\textbf{CRing}^{op} \to \textbf{Top}$ sending R to $\text{Spec}(R)$, which consists of prime ideals under the Zariski topology, in which closed subsets are all the prime ideals containing a certain ideal. It's contravariant because of a result in commutative algebra guaranteeing that pre-images of prime ideals are prime ideals, though the images may not be.

Proposition 3.8. *Functors preserve isomorphisms.*

Proof.

$$\begin{aligned} F(f)F(f^{-1}) &= F(f \circ f^{-1}) \\ &= F(\mathbb{1}) \\ &= \mathbb{1} \end{aligned}$$

and likewise in the other direction. □

Example 3.9. Recall that associated to a group G we have the single object category BG. What does a functor $BG \to \mathcal{C}$ look like? Functors preserve automorphisms, so BG gets sent to a single object in $\mathcal{C}$ and a collection of automorphisms on that object. So a functor $BG \to \mathcal{C}$ is equivalent to a G-object in $\mathcal{C}$ (i.e. an object with a G-action).

When $\mathcal{C}$=**Set**, the data of a functor is a G-set. When $\mathcal{C} = \textbf{Vect}_k$, the data of a functor is a representation. And when $\mathcal{C}$ = **Top**, it's a G-space.

Though we just learned that functors preserve isomorphisms, they do not in general preserve monomorphisms and epimorphisms. But they *do* preserve split monomorphisms and epimorphisms (i.e. one-sided inverses).

Definition 3.10. For $\mathcal{C}$ a locally small category, there are two functors from $\mathcal{C}$ to **Set**. One is covariant, and is defined $(-) \mapsto \text{Hom}(c, -)$ for fixed c. Another, contravariant, is defined $(-) \mapsto \text{Hom}(,c)$.

Definition 3.11. Given categories $\mathcal{C}, \mathcal{D}$, their product category $\mathcal{C} \times \mathcal{D}$ is a category whose objects are pairs (c, d) and whose morphisms are pairs of appropriate morphisms in each 'coordinate'. Bifunctors are functors out of product categories.

4. Lecture 4 – February 15, 2019

Last time we learned that a functor is a map between categories.

Definition 4.1. **Cat** is the the category whose objects are small categories. It's not small, but it is locally small.

Subcategories of **Cat** include **Set**, **Grp**, the category of monoids, and the category of groupoids.

Definition 4.2. CAT is the category whose objects are locally small categories. It's not even locally small.

As you may expect, there's an inclusion functor $\mathbf{Cat} \hookrightarrow \mathbf{CAT}$.

Definition 4.3. An isomorphism of categories $F : \mathcal{C} \to \mathcal{D}$ is a functor which admits an inverse $G : \mathcal{D} \to \mathcal{C}$ such that $GF = \mathbb{1}_{\mathcal{C}}$ and $FG = \mathbb{1}_{\mathcal{D}}$.

Example 4.4. $(-)^{op} : \mathbf{CAT} \to \mathbf{CAT}$ is an automorphism of **CAT**.

Note that for R a ring, R^{op} consists of the asme underlying elements with opposite multiplication. It follows that any left R-module is equivalent to a right R-module, and $\mathbf{Mod}_R \cong {R^{op}}_{\mathbf{Mod}}$.

Though we've presented a few examples, this notion of isomorphism is almost always too restrictive in practice. Intuitively, what we've said is that two objects in a category are equivalent if they're outright equal, but we'd like to say they're equivalent if they're isomorphic. Historically, the development of category theory had somewhat backward motivations, in that people only cared about categories insofar as they cared about natural transformations.

Example 4.5. Take V to be a finite dimensiona lvector space over K, and $V^* = \mathrm{Hom}(V, K)$. We've seen in linear algebra that $V \cong V^*$. To see why, we select a basis for V: $e_1, \ldots, e_n$. Now take e_1^* to be the linear functional sending e_1 to 1 and everything else to 0. The e_i^* are then a basis for V^*, and the isomorphisms from V to V^* sends e_j to e_j^*.

Now let's look at $(V^*)^* := \mathrm{Hom}(V^*, K)$. We already know that $(V^*)^* \cong V^* \cong V$, but there's a choice-free way to exhibit the long isomorphism from V to $(V^*)^*$. Consider the map sending v to $ev_v : f \mapsto f(v)$. It's an isomorphism. The point of this exercise is that this isomorphism is, if only informally, quite natural. The first isomorphism, from V to V^*, forced us to make a choice and get our hands dirty.

"Not natural means it's not natural." - Danny.

Definition 4.6. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\alpha : F \to G$ consists of a morphism $\alpha_c : Fc \to Gc$ for each $c \in \mathcal{C}$, called the component of the natural transformation, such that $\alpha_{c'} \circ Ff = Gf \circ \alpha_c$.

$$\begin{array}{ccc} Fc & \xrightarrow{\alpha_c} & Gc \\ \big\downarrow{\scriptstyle Ff} & & \big\downarrow{\scriptstyle Gf} \\ Fc' & \xrightarrow{\alpha_{c'}} & Gc' \end{array}$$

Definition 4.7. A natural isomorphism is a natural transformation whose components are all isomorphisms.

Example 4.8. Let's think back to our evaluation map $ev : V \to V^{**}$ that sends vectors in V to the map sending an element of V^* to its output on v. We claim that it's a natural transformation between the identity on Vect_k and the double dual on Vect_k. This amounts to showing that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{ev} & V^{**} \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi^{**}} \\ W & \xrightarrow{ev} & W^{**} \end{array}$$

To see why there's couldn't be a natural transformation between the identity and $(-)^*$, note that the $(-)^*$ is contravariant.

People usually end up saying that functors are natural if they admit natural transformations with the identity.

Example 4.9. We've seen that a functor $BG \to \mathcal{C}$ is an object in C with a G-action. So a map from two functors is a G-equivariant map between objects in $\mathcal{C}$. As expected, a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

"Do you like this? Do you like it as much as I do?" - Danny.

Exercises 1.4(i), 1.4(ii), 1.4(iv)

5. Lecture 5 – February 20, 2019

"The gloves are coming off." - Danny

Today we're going to talk about equivalence of categories. We've spoken briefly about the notion of homotopy in algebraic homotopy - informally, a homotopy between maps $f, g : X \to Y$ on topological spaces corresponds to the statement that the maps can be continuously deformed into each other. Formally, a homotopy is a continuous map $h : X \times [0,1] \to Y$ with $X \times \{0\} = f$ and $X \times \{1\} = g$.

This definition looks somewhat similar to what we saw when considering natural transformations, in that you may be tempted to think of a natural transformation as a homotopy between functors - one important difference is that the property of admitting a homotopy is symmetric while that of admitting a natural transformation is not. Natural isomorphisms are in this sense better analogs for homotopy for functors, as they admit natural transformations in the opposite direction.

Today it will be useful to consider the category $\mathbf{1}$ of one object 0 and the category $\mathbf{2}$ with objects 0, 1 and one non-identity morphisms from 0 to 1.

Proposition 5.1. *Suppose α is a natural transformation between $F, G : \mathcal{C} \to \mathcal{D}$. Then α corresponds bijectively to a functor $\mathcal{C} \times \mathbf{2} \to \mathcal{D}$.*

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{i_0} & \mathcal{C} \times \mathbf{2} & \xleftarrow{i_1} & \mathcal{C} \\ & \searrow^{F} & \downarrow^{H} & \swarrow_{G} & \\ & & \mathcal{D} & & \end{array}$$

Recall that a natural isomorphism is a natural transformation with an inverse natural transformations. Now we'll look at the category $\mathbb{I}$, which is the category $\mathbf{2}$ with a morphism from 1 to 0. We have a similar proposition.

Proposition 5.2. *The data of a natural isomorphism between $F, G : \mathcal{C} \to \mathcal{D}$ is equivalent to a functor H with*

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{i_0} & \mathcal{C} \times \mathbb{I} & \xleftarrow{i_1} & \mathcal{C} \\ & \searrow^{F} & \downarrow^{H} & \swarrow_{G} & \\ & & \mathcal{D} & & \end{array}$$

Recall that categories are isomorphic if they admit functors which compose to the identity in each direction, and that this is really too strong a property for us to be interested in. Now we'll look at an alternative property.

Definition 5.3. Categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $GF \cong \mathbb{1}_{\mathcal{C}}$ and $FG \cong \mathbb{1}_{\mathcal{D}}$.

Categories which are equivalent are written $\mathcal{C} \simeq \mathcal{D}$, while isomorphic categories are written $\mathcal{C} \cong \mathcal{D}$. As you'd expect, equivalence of categories is an equivalence relations - the only non-trivial condition to verify is transitivity.

Example 5.4. Recall that Mat_K is the category whose objects are the natural numbers and where $\mathrm{Hom}(n, m)$ consists of $n \times m$ matrices over K. Composition is defined by multiplying matrices in opposite order. Vect_K^{fd} is the category of finite-dimensional vector spaces over K while $\mathrm{Vect}_K^{\mathrm{basis}}$ is the category of finite-dimensional vector spaces over K with a chosen basis.

$U : \mathrm{Vect}_K^{\mathrm{basis}} \to \mathrm{Vect}_K^{fd}$ forgets the basis, while $C : \mathrm{Vect}_K^{fd} \to \mathrm{Vect}_K^{\mathrm{basis}}$ selects a basis for vector spaces.

$K^{(-)} : \mathrm{Mat}_K \to \mathrm{Vect}_K^{\mathrm{basis}}$ sends n to K^n with standard basis. Finally, $H : \mathrm{Vect}_k^{\mathrm{basis}} \to \mathrm{Mat}_K$ sends a vector space to its dimension.

Theorem 5.5. *The above functors produce equivalences between the three categories.*

Fortunately, there's an easier way of proving this than checking manually that all the appropriate compositions admit natural isomorphisms to the identity.

Definition 5.6. A functor $F : \mathcal{C} \to \mathcal{D}$ is full if for each $x, y \in \mathcal{C}$, the map $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$ is surjective. It is faithful if this map always injects.

Notice that these are both local conditions. It's in fact possible for a functor to surject on objects and be fully faithful, yet fail to be an isomorphism of categories - in particular, it could still fail to inject on objects.

Definition 5.7. A functor $F : \mathcal{C} \to \mathcal{D}$ is essentially surjective on objects if for each $d \in \mathcal{D}$, $d \cong Fc$ for some $c \in \mathcal{C}$.

Theorem 5.8. *A functor defines an equivalence between categories if and only if it is full, faithful, and essentially surjective on objects*[1].

Definition 5.9. A faithful functor which injects on objects is an embedding.

Embeddings identify their domains as subcategories of codomains. A full embedding identifies its domain as a full subcategory of its codomain.

6. Lecture 6 – February 27, 2019

Recall that categories $\mathcal{C}, \mathcal{D}$ are equivalent if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ with GF and FG admitting natural transformations to the identity.

We also learned about less tedious ways to detect equivalences of categories, via the local properties of fullness and faithfulness and via the global property of essential surjectiveness (i.e. surjectiveness up to isomorphism). In particular, we saw the following theorem:

Theorem 6.1. *A functor $F : \mathcal{C} \to \mathcal{D}$ induces an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. First a lemma: suppose $f : a \to b$ is a morphisms in a category and $a \cong a'$, $b \cong b'$ in this category. Then there's a unique morphisms $f' : a' \to b'$ such that all the following diagrams commute:

$$\begin{array}{ccc} a & \xleftarrow{\ \cong\ } & a' \\ \downarrow f & & \downarrow f' \\ b & \xrightarrow{\ \cong\ } & b' \end{array}$$

And the three other diagrams given by changing the directions of the isomorphisms between a, a' and b, b'. The proof is to define f' using this diagram and then check that the diagrams commute.

Back to the proof - suppose F induces an equivalence, meaning there's an inverse functor $G : \mathcal{D} \to \mathcal{C}$ so that there are natural isomorphisms $GF \cong \mathbf{1}_{\mathcal{C}}$ and $FG \cong \mathbf{1}_{\mathcal{D}}$. To see that F is essentially surjective, fix $d \in \mathcal{D}$. We'd like to show that $F(Gd) \cong d$, but this follows from the fact that $FG \cong \mathbf{1}_D$.

[1]The backward direction requires use of the Axiom of Choice, but we won't worry about that here.

To show faithfulness, suppose that $Ff = Fg$, for $f, g \in \text{Hom}(c, c')$. The idea is to show that f and g both satisfy the diagrams from our lemma, and it'll follow that they must be the same. Since $GFf = GFg$, and $\mathbf{1}_{\mathcal{C}} \cong GF$, we have

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ \Big\downarrow {\scriptstyle f \text{ or } g} & & \Big\downarrow {\scriptstyle GFf} \\ c' & \xrightarrow{\eta_{c'}} & GFc' \end{array}$$

Since both f and g make this diagram commute (and the other three), by our previous lemma they must coincide. Note that, by symmetry, this also means that G is faithful.

Now fix $k \in \text{Hom}(Fc, Fc')$. Then $Gk \in \text{Hom}(GFc, GFc')$. By our natural isomorphism, we have the diagram:

$$\begin{array}{ccc} c & \xrightarrow{\eta_c} & GFc \\ \Big\downarrow {\scriptstyle \exists! h} & & \Big\downarrow {\scriptstyle GFk} \\ c' & \xrightarrow{\eta_{c'}} & GFc' \end{array}$$

The unique existence of such an h follows again from our lemma. Now note that replacing Gk with GFh still makes the diagram commute, because $GF \cong \mathbf{1}_{\mathcal{C}}$. So $Gk = GFh$, and because G is faithful, we're done.

Now suppose that $F : \mathcal{C} \to \mathcal{D}$ is fully faithful and essentially surjective. First we need to construct $G : \mathcal{D} \to \mathcal{C}$ - given $d \in D$, send it to one of the objects in $\mathcal{C}$ whose image under F is isomorphic to d. Then we have

$$\begin{array}{ccc} FGd & \xrightarrow{\varepsilon_d} & d \\ \Big\downarrow {\scriptstyle \exists!} & & \Big\downarrow {\scriptstyle \ell} \\ FGd' & \xrightarrow{\varepsilon_{d'}} & d' \end{array}$$

The unique map making this diagram commute is where $\ell : d \to d'$ gets sent. One must show that G is in fact a functor (meaning it sends identities to identities and respect composition), but it then follows immediately that $FG \cong \mathbf{1}_{\mathcal{D}}$. It remains only to show that $GF \cong \mathbf{1}_{\mathcal{C}}$. This involves writing down diagrams and doing some chasing :) □

Note that this proof was not too hard given familiarity with definitions - this is a common theme in category theory. Precise definitions make lots of proofs surprisingly easy.

Definition 6.2. A category is <u>connected</u> if any pair of objects is connected by a zig-zag of morphisms.

In the above definition, the category is thought of as a graph whose vertices are objects and whose morphisms form undirected edges.

Proposition 6.3. *Any connected groupoid is equivalent to the automorphism group of any of its objects.*

Proof. There's an inclusion of categories F from the automorphism group of an object to the connected groupoid. F is essentially surjective because in a connected groupoid, all the objects are isomorphic to each other. It's also trivially fully faithful, so it's an equivalence of categories. $\square$

Definition 6.4. A category $\mathcal{C}$ is skeletal if it contains only one object in each isomorphism class.

Given a category $\mathcal{C}$, $sk(\mathcal{C})$ is the unique skeletal category that is equivalent to $\mathcal{C}$. Turning a connected groupoid into its skeleton is what we did in the previous proposition.

Notice that for skeletal categories, an equivalence of categories is an isomorphism of categories. Additionally, $C \simeq D \iff sk(\mathcal{C}) \cong sk(\mathcal{D})$.

7. LECTURE 7 – MARCH 8, 2019

Definition 7.1. An object $i \in \mathcal{C}$ is initial if $\mathrm{Hom}(i, i = c)$ consists of exactly one element for all $c \in \mathcal{C}$.

As usual, a definition is accompanied by its dual definition.

Definition 7.2. An object $t \in C$ is terminal if $\mathrm{Hom}(c, t)$ consists of exactly one element for all $c \in \mathcal{C}$.

Example 7.3. In **Set**, $\emptyset$ is the initial object and $\{\mathrm{pt}\}$ is the terminal object. In **Top**, the initial object is the empty space and the terminal object is the space with one point. In $\mathbf{Mod}_R$, the initial and terminal objects are both the 0 module. In **Field**, there's no initial object or terminal object, because there are no field homomorphisms between fields of different characteristic. In **Cat**, the initial object is the empty category and the terminal object is $\mathbb{1}$, the category with one object.

We're done with Chapter 1 and we're ready to discuss Chapter 2, which is about the Yoneda Lemma. First, we'll cast the definitions of initial and terminal objects in terms of induced functors.

Definition 7.4. An object $c \in \mathcal{C}$

(1) is initial if the functor $\mathrm{Hom}(c, -) : \mathcal{C} \to \mathbf{Set}$ is naturally isomorphic to the constant functor $* : \mathcal{C} \to \mathbf{Set}$ that sends all $x \in \mathcal{C}$ to the one element set.
(2) is terminal if the functor $\mathrm{Hom}(-, c) : \mathcal{C}^{op} \to \mathbf{Set}$ is naturally isomorphic to the functor $* : \mathcal{C}^{op} \to \mathbf{Set}$.

This demonstrates a theme of category

Definition 7.5. A functor $F : \mathcal{C} \to \mathbf{Set}$ is representable if there is an object $c \in \mathcal{C}$ and a natural isomorphism $F \simeq \mathrm{Hom}(c, -)$ or $F \simeq \mathrm{Hom}(-, c)$. One then says that F is represented by c.

This is Emily's definition of representability, but it's often said that a natural isomorphism to $\mathrm{Hom}(c, -)$ makes F co-representable while a natural isomorphism to $\mathrm{Hom}(,c)$ makes it representable.

Definition 7.6. A representation for F is an object $c \in \mathcal{C}$ together with a choice of natural isomorphism $F \simeq \text{Hom}(c, -)$ or $F \simeq \text{Hom}(-, c)$.

Universal properties say things about either $\text{Hom}(X, -)$ or $\text{Hom}(-, X)$.
"Do you know what Mean Girls is?" - Danny.

Example 7.7.
(1) The identity functor from **Set** to **Set** is represented by the singleton **1**, because $\text{Hom}(\mathbf{1}, X) \cong X$.
(2) The forgetful functor **Grp** $\to$ **Set** is represented by $\mathbb{Z}$, since $\text{Hom}(\mathbb{Z}, G) \cong G$.
(3) The forget functor **Ring** $\to$ **Set** is represented by $\mathbb{Z}[x]$, because a map out of $\mathbb{Z}[x]$ is determined by where x goes, and it can go anywhere.
(4) The forgetful functor from $\mathbf{Mod}_R \to$ **Set** is represented by R.
(5) The functor from **Grp** to **Set** which sends G to the set of its n-tuples is represented by the free abelian group on n generators.
(6) The functor from **Ring** to **Set** sending a ring to the collection of its units is represented by $\mathbb{Z}[x, x^{-1}]$.
(7) The functor Hom : **Cat** $\to$ **Set** sending a category to the set of its morphisms is represented by the category **2**, which has two objects and one non-identity morphism between them.
(8) The functor Iso : **Cat** $\to$ **Set** sending a category to the collection of its isomorphisms is represented by the category $\mathbb{I}$ which has two objects and a non-identity morphisms in each direction (which, of course, are obligated to be inverses).
(9) The functor Path : **Top** $\to$ **Set** sending a topological space to the collection of its paths is represented by $[0, 1]$.
(10) The functor Loop : **Top** $\to$ **Set** sending a topological space to the collection of its loops is represented by S^1.

Eilenberg-MacLane space: $\text{Hom}_{hTop}(X, K(A, n)) = H^n(X; A)$. Yoneda lemma turns natural transformations between functors into maps between the objects representing them

8. Lecture 8 – March 13, 2019

Back in the 40's, when Eilenberg and MacLane invented category theory, a Japanese mathematician and computer scientist called Nobuo Yoneda was visiting Paris.

MacLane wanted to meet Yoneda, who said he was short on time and needed to meet in a train station. So they're speaking in a caf about Yoneda's lemma, and the train arrives before they've finished speaking. Yoneda got on the train, and MacLane followed him without a ticket - MacLane was able to learn about the Yoneda Lemma, which he loved and then advertised.

"If you had a time machine to go back 60 years ago, you could be a pretty big shot in category theory."

The Yoneda Lemma is about functors from categories to sets - functors from $\mathcal{C}^{op} \to$ **Set** are called presheaves. We've already learned that functors from $\mathcal{C} \to$ **Set** which can be witnessed as $\mathcal{C}(c,-)$ or $\mathcal{C}(-,c)$ are representable. This method of producing functors provides an embedding of $\mathcal{C}$ in its functors to **Set**.

A natural question asks which kinds of functors are representable, and how natural transformations from representable functors can be characterized.

Lemma 8.1 (Yoneda Lemma)**.** *For $\mathcal{C}$ a locally small category and $F : \mathcal{C} \to$ **Set**, there exists a natural isomorphism*

$$\mathrm{Hom}(\mathrm{Hom}(c,-),F) \cong Fc$$

Proof. We can construct a map $\Phi : \mathrm{Hom}(\mathrm{Hom}(c,-),F) \to Fc$ by sending α to $\alpha_c(\mathbb{1}_c)$. In the other direction, agh idk. Somehow functors are specified by where they send one thing. Then need to check that this bijection is natural. □

Example 8.2.

(a) Take $\mathcal{C}$ to be the category

$$1 \to 2 \to 3 \to 4 \to \dots$$

and $F : \mathcal{C} \to$ **Set** to be some functor. Fixing a $k \in \mathcal{C}$, we're interested in natural transformations $\mathcal{C}(k,-) \implies F$. Taking $k = 4$, we have $\mathrm{Hom}(4,x) = \emptyset$ if $x < 4$ and $\mathrm{Hom}(4,x) = *$ otherwise. A natural transformation from $\mathrm{Hom}(4,-)$ to F consists of the following commutative diagram:

$$\begin{array}{l} \emptyset \to \emptyset \to \emptyset \to * \to * \to \dots \\ \downarrow \quad\ \downarrow \quad\ \downarrow \\ F_1 \to F_2 \to F_3 \to F_4 \to F_5 \to \dots \end{array}$$

(b) Now consider the category BG - we've mentioned that a functor $BG \to$ **Set** is equivalent to a left G-set. A natural transformation between $\mathrm{Hom}(*,-)$ and $X : BG \to$ **Set** corresponds to a G-equivariant map out of G (with the structure of a G-action). And a G-equivariant map out of G is determined by its output on any single element.

One interesting result of the Yoneda Lemma is that it gives rise to the Yoneda embedding (taking the dual, it in fact gives rise to two Yoneda embeddings). In particular, there's an embedding

$$\mathcal{C} \hookrightarrow \mathbf{Set}^{\mathcal{C}^{op}}$$

And another

$$\mathcal{C}^{op} \hookrightarrow \mathbf{Set}^{C}$$

It turns out that these embeddings are full and faithful. One result of this is that a natural transformation $\mathrm{Hom}(d,-) \implies \mathrm{Hom}(c,-)$ is equivalent to a morphism $c \to d$. This is sometimes presented as the statement of Yoneda's lemma, and is its most common use.

Corollary 8.3 (Cayley's Theorem)**.** *Any group G is isomorphic to a subgroup of a permutation group.*

10. Lecture 10 – March 29, 2019

Last time we talked about adjoint functors. For the categories A, B, the functor $G : B \to A$ is adjoint to $F : A \to B$ if $\text{Hom}_B(F(a), b) \cong \text{Hom}_A(a, G(b))$. Note that this is a natural isomorphism. Today, we're going to define adjoint functors in terms of unit and counit maps.

For adjoint functors, there are two special maps. First suppose that $b = F(a)$, meaning we're considering an endomorphism on $F(a)$. In fact, consider the identity on $F(a)$. Since G is adjoint to F, G sends this to a map $a \to GF(a)$, and this map is natural in a. So we've arrived at a natural transformation $\eta : \mathbb{1}_A \to GF$. Dually, there's a natural transformation $\epsilon : FG \to \mathbb{1}_B$. η is called a unit map, while ϵ is a counit map.

Example 10.1. The free functor $F : \mathbf{Set} \to \mathbf{Vect}_k$ is adjoint to the forgetful functor $U : \mathbf{Vect}_k \to \mathbf{Set}$. By our previous map, there should be a unit map $\eta : \mathbb{1}_{\mathbf{Set}} \to UF$ and a counit map $\epsilon : FU \to \mathbb{1}_{\mathbf{Set}}$. Notice that η sends an element to itself, though it lies in a larger set (the set of the vector space with basis the original set containing our element). On the other hand, the counit map has domain which is the free vector space on the set of the original vector space (it's huge). It sends $\sum_{v \in V} \lambda_v v$ to itself as an evaluated sum in V.

Lemma 10.2 (Triangle Identities)**.** *This diagram commutes:*

$$\begin{array}{ccc} F(a) & \xrightarrow{F(\eta_A)} & FG(F(a)) \\ & \searrow^{\mathbb{1}_{F(a)}} & \downarrow \eta_{F(a)} \\ & & F(a) \end{array}$$

A dual diagram holds when one replaces a with b and F with G.

Proof. Consider the transpose of $\eta_A : A \to GF(A)$; it corresponds to the identity on $F(A)$. It may seem silly, but consider the transpose of the composition $\mathbb{1}_{GF(A)} \circ \eta_A$. First transposing the identity map, and using adjoint-ness of G, F, this corresponds to the counit map $\epsilon_{F(A)} : FGF(A) \to F(A)$. Now applying F to the first map in our composition, we arrive at

$$F(A) \xrightarrow{F(\eta_A)} FGF(A) \xrightarrow{\epsilon_{F(A)}} F(A)$$

Since this is the identity - as the transpose of η_A is the identity - we've shown that the diagram commutes. □

It turns out that the data of these functors being adjoint is equivalent to the statement that these unit and counit maps obey the above triangle identities. So today's slogan is that the unit and the counit determine the whole adjunction.

Lemma 10.3. *Consider another adjunction $F : A \to B$ and $G : B \to A$. Then by the previous lemma, the transpose of $g : F(A) \to B$ is $\overline{g} = G(g) \circ \eta_A : A \to G(B)$.*

Theorem 10.4. *Given functors $F : A \to B$, $G : B \to A$, there's a one-to-one correspondence between:*

(a) Adjunctions $F \dashv G$

(b) Pairs $\eta : \mathbb{1}_A \to GF$, $\epsilon : \mathbb{1}_B \to FG$ of natural transformations satisfying the triangle identities.

Proof. This formalizes our previous statement that adjunctions are totally specified by their unit and co-unit maps. The map $(a) \to (b)$ is obvious. In the other direction, given $F \dashv G$, the adjuction $\mathrm{Hom}_B(F(A), B) \cong \mathrm{Hom}_A(A, G(B))$ must be unique if it exists. To see that it exists, take a map $g : F(A) \to B$. We send it to $G(g) \circ \eta_A$. On the other hand, given $f : A \to G(B)$, we send it to $\epsilon_B \circ F(f)$. It remains to show that these maps are inverses and that they're natural. The maps are natural because they are. That they're inverses is left as an exercise, and is not too illuminating. ☐

Example 10.5. Given ordered sets A, B, order-preserving functions $f : A \to B$, $g : B \to A$ are adjoint iff $f(a) \leq b \iff a \leq g(b)$.

Fix X a topological space and $\mathcal{P}(X)$ - its power set - and $\mathcal{C}(X)$ - its closed subsets. There's an inclusion $\mathcal{C}(X) \to \mathcal{P}(X)$ and a map $\mathcal{P}(X) \to \mathcal{C}(X)$ which takes closures. The closure functor is left-adjoint to the inclusion functor, because $Cl(A) \subseteq B \iff A \subseteq \mathrm{int}(A)$.

11. Lecture 11 – April 3, 2019

We've been talking about left-adjoint pairs $F \dashv G$ for $F : A \to B$ and $G : B \to A$, where $\mathrm{Hom}_B(F(a), b) \cong \mathrm{Hom}_A(a, G(b))$. We also learned about the unit map $\eta_a : a \to GF(a)$, which is associated to the identity on $F(a)$, and the counit map $\epsilon_a : FG(a) \to a$ associated to the identity on $G(a)$.

We also learned about the correspondence between adjoint functors and the (co)unit maps. Today, we'll obtain another characterization of adjoint functors which uses initial objects. In particular, that $F \dashv G$ is equivalent to the statement that a certain category contains an initial object.

We've talked about how the free and forgetful functors are adjoint on $\mathbf{Vect}_k$ and **Set**.

Definition 11.1. Given functors $P : \mathcal{A} \to \mathcal{C}$ and $Q : \beta \to \mathcal{C}$, one can construct the comma category (P $\implies$ Q) whose objects are triples $(A, h : P(A) \to Q(B), B)$ and whose morphisms $\mathrm{Hom}((A, h, B), (A', h', B'))$ are pairs of morphisms $f : A \to A'$, $g : B \to B'$ which make the following diagram commute:

Remark 11.2. There are 'projection' functors from (P $\implies$ Q) to β and $\mathcal{A}$. This gives rise to two functors from $(P \implies Q)$ to $\mathcal{C}$: projecting onto β and applying Q, and projecting onto $\mathcal{A}$ and applying P. These functors admit a natural transformation, which explains the notation $(P \implies Q)$!

Example 11.3. Slice categories are comma categories. Recall that the slice category $\mathcal{A}/A$ has objects which are morphisms in $\mathcal{A}$, $h : X \to A$, and has morphisms which are maps $f : X \to X'$ making the triangle commute, i.e. $h' \circ f = h$. The slice ceategory is precisely $(\mathbb{1}_\mathcal{A} \implies A)$, where $A \in \mathcal{A}$ is though of as a functor from the category $\mathbb{1}$ to $\mathcal{A}$.

There's also the coslice category $A/\mathcal{A}$ whose objects are $h : A \to X$ and whose morphisms are $f : X \to X'$ such that $f \circ h = h'$. This category is $(A \implies \mathbb{1}_\mathcal{A})$.

Now we'll consider the setup

$$\begin{array}{ccc} & & \beta \\ & & \big\downarrow{\scriptstyle G} \\ \mathbb{1} & \xrightarrow{A} & \mathcal{A} \end{array}$$

, keeping in mind that our goal is to make a connection between comma categories and adjunctions.

Lemma 11.4. *Say we have an adjunction $F : \mathcal{A} \to \beta$, $G : \beta \to \mathcal{A}$. We have a unit map $\eta_A : A \to GF(A)$. This is in fact an object in the previous comma category - $(F(A) \in \beta, \eta_A) \in (A \implies G)$. Moreoever, this object is initial in $(A \implies G)$.*

Proof. We'd like to show $\mathrm{Hom}((F(A), \eta_A), (B, F : A \to G(B)))$ consists of exactly one map. This amounts to showing that there exists exactly one $G(q)$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GF(A) \\ & \searrow^{f} & \big\downarrow{\scriptstyle G(q)} \\ & & G(B) \end{array}$$

We know that a q making the diagram commute would then have $\overline{q} = G(q) \circ \eta_A = f$. Taking transposes again, $q = \overline{\overline{q}} = \overline{f}$. So $q = \overline{f}$. ☐

Theorem 11.5. *Given functors $F : \mathcal{A} \to \beta$, $G : \beta \to \mathcal{A}$, there's a one-to-one correspondence between:*

(1) Adjunctions between F and G.
(2) Natural transformations $\eta : \mathbb{1}_\mathcal{A} \implies GF$ with $\eta_A : A \to GF(A)$ is initial in $(A \implies G)$ for all $A \in \mathcal{A}$.

Proof. We've already shown $(1) \implies (2)$. The other direction is in Leinster, and is a bit ugly. ☐

Corollary 11.6. *$G : \beta \to A$ has a left adjoint if and only if for any $A \in \mathcal{A}$, the comma category $(A \implies G)$ has an initial object.*

Proof. We've already shown the forward direction. In the reverse direction, we'd like to construct F such that $F \dashv G$. ☐

This isn't really coincidence - comma categories are defined so that their initial objects contain information about adjoints.

12. Lecture 12 – April 5, 2019

The three main topics in an introductory category theory class are adjoints, representability, and limits/colimits. You can do them in more or less any order, and we've decided to first cover representability, then adjoints, and now limits/colimits. At a high leve, adjoints deal with functors between categories, representability seeks to understand a category by considering its functors to **Set**, and limits/colimits seek to understand a category by only considering things in the category itself.

We'll look at three kinds of limits today - the first are products. For sets X, Y, the product $X \times Y$ is the set whose elements are pairs of elements in X and elements in Y. The universal property of the product here is that a map $W \to X \times Y$ corresponds to a map $W \to X$ and another $W \to Y$.

Definition 12.1. Take $X, Y \in \mathcal{A}$. A product of X and Y is an object $P \in \mathcal{A}$ equipped with maps $p_1 : P \to X$ and $p_2 : P \to Y$ such that [UNIVERSAL PROPERTY OF PRODUCT]

Products are examples of limits!

Remark 12.2. Products don't always exist. One example is a category in which X or Y don't receive any morphisms, meaning there can't be projection maps $p_i : X \times Y \to X, Y$.

Proposition 12.3. *When products exist, they're unique up to isomorphism.*

Proof. Plug them both into the universal property. □

Example 12.4.
(a) In **Top**, the product $X \times Y$ is the set $X \times Y$ equipped with the product topology. As you might expect, the product topology is defined so that this be the case. In particular, it's designed so that $A \to X \times Y$ be continuous if and only if the induced maps $A \to X$ and $A \to Y$ are continuous.
(b) In $\mathbf{Vect}_K$, the product of U, V is the direct sum $U \oplus V$, as it satisfies the universal property.
(c) Take the category $\mathbb{R}$ induced by its total order, i.e. $\mathrm{Hom}(a, b) = *$ if $a < b$ and it's empty otherwise. In this case, $X \times Y = \min\{X, Y\}$.
(d) Take the category $\mathbb{N}$ induced by the partial order of divisibility. Then $a \times b = \gcd(a, b)$

Definition 12.5. For $\mathcal{A}$ a category, I a set, the product of $(X_i)_{i \in I}$ is an object $P \in \mathcal{A}$ equipped with projection maps $p_i : P \to X_i$ such that maps $W \to X_i$ collectively factor uniquely through P and its p_i.

As a special case, when the indexing set is empty, the product is a terminal object in the category. In fact, limits are alwasy related to terminal objects in certain categories, and colimits are always related to initial objects in certain categories. The second example of a limit we'll look at today is the equalizer.

Definition 12.6. For $X, Y \in \mathcal{A}$ and $s, t : X \to Y$, their equalizer is an object $E \in \mathcal{A}$ equipped with a map $i : E \to X$ such that $s \circ i = t \circ i$ and such that for any $f : A \to X$, there is a unique $\overline{f} : A \to E$ such that $i \circ \overline{f} = f$.

The final setup, with $i : E \to X$ and $s, t : X \to Y$ satisfying $s \circ i = t \circ i$, is called a fork. The equalizer is a terminal object in the category of forks.

Remark 12.7. Same as last time - equalizers don't always exist, but when they do they're unique up to isomorphism.

Example 12.8.
(a) In **Set**, the equalizer of $s, t : X \to Y$ is $E = \{x \in X | s(x) = t(x)\}$ equipped with the inclusion to X.
(b) In **Top**, the equalizer of $s, t : X \to Y$ is the equalizer from **Set** equipped with the subspace topology (which makes the inclusion continuous).
(c) In **Grp**, the equalizer of $\phi, \epsilon : G \to H$ is $\ker \phi$.
(d) In $\mathbf{Vect}_K$, the equalizer of $s, t : V \to W$ is $\ker(s - t) \subseteq V$.

To get the equalizer of a collection of objects, you can take the equalizer of their product, which is pretty cool. The last example of a limit we'll look at today is the pullback.

Definition 12.9. Given X, Y, Z with $t : Y \to Z$ and $s : X \to Z$, the pullback is an object P equipped with maps $P_1 : P \to X$ and $P_2 : P \to Y$ such that maps $f_1 : A \to X$ and $f_2 : A \to Y$ factor uniquely through P.

When Z is a terminal object, the pullback is the product.

Example 12.10.
(1) When Z is a terminal object, the pullback is the product.
(2) In **Set**, the pullback of $s : X \to Z$ and $t : Y \to Z$ is the set $P = \{(x, y) \in X \times Y | s(x) = t(y)\}$ equipped with projections onto X, Y.
(3) Again in **Set**, for $f : X \to Y$ and $i : Y' \to Y$, the pullback is $f^{-1}(Y')$ equipped with the inclusion to X and $f : f^{-1}(Y') \to Y$.